The Scattering Matrix

At **"low"** frequencies, we can completely characterize a **linear** device or network using an **impedance** matrix, which relates the currents and voltages at **each** device terminal to the currents and voltages at **all** other terminals.

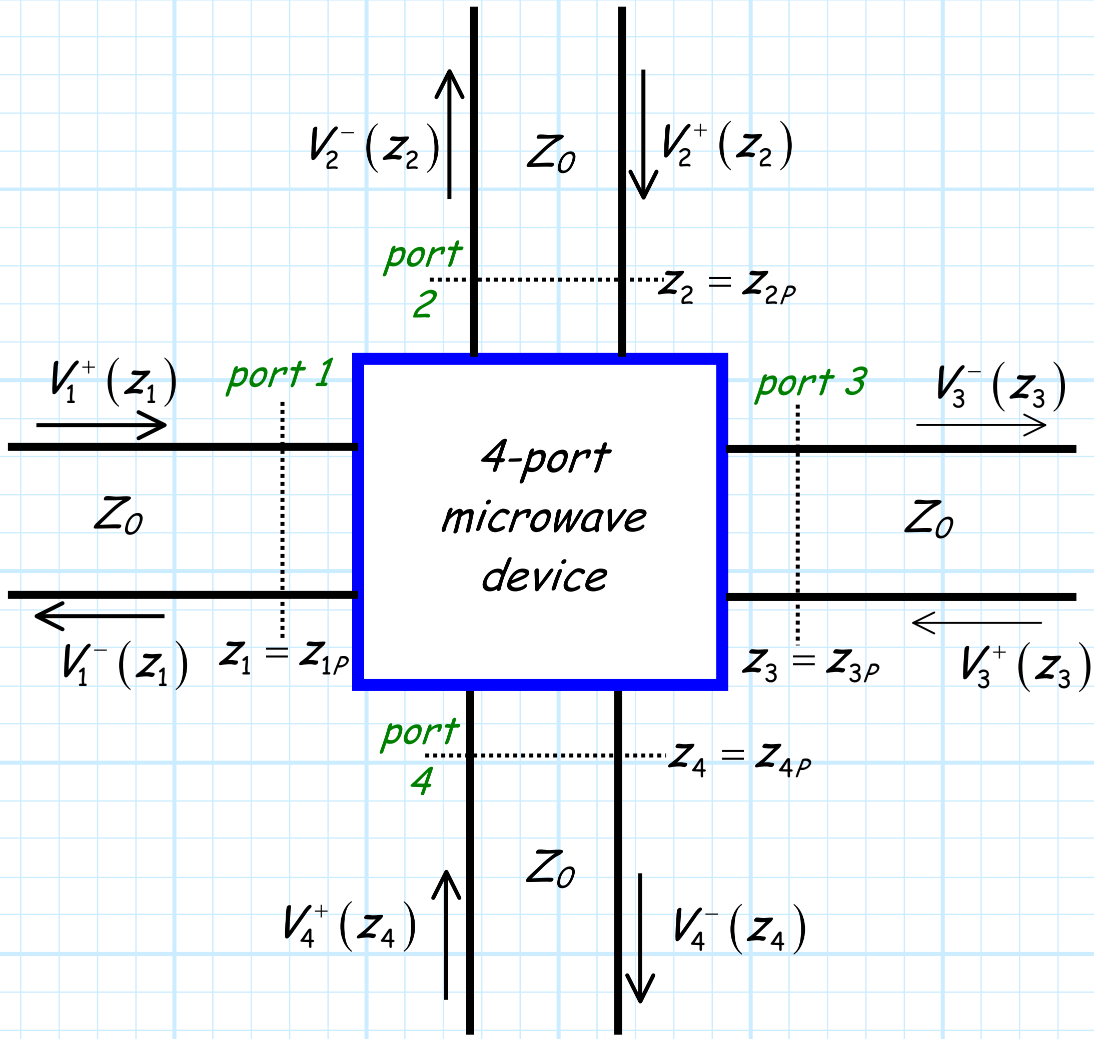
But, at microwave frequencies, it is **difficult** to measure total currents and voltages!

* Instead, we can measure the **magnitude** and **phase** of each of the two transmission line **waves** $V^+(z)$ and $V^-(z)$.

* In other words, we can determine the relationship between the incident and reflected wave at **each** device terminal to the incident and reflected waves at **all** other terminals.

These relationships are completely represented by the **scattering matrix**. It **completely** describes the behavior of a linear, multi-port device at a **given frequency** ω.

Consider the **4-port** microwave device shown below:



Note in this example, there are four **identical** transmission lines connected to the same "box". Inside this box there may be a very **simple** linear device/circuit, **or** it might contain a very large and **complex** linear microwave system.

→ Either way, the "box" can be fully characterized by its **scattering parameters**!

First, note that each transmission line has a specific **location** that effectively defines the **input** to the device (i.e., z_{1P}, z_{2P}, z_{3P}, z_{4P}). These often arbitrary positions are known as the **port** locations, or port **planes** of the device.

Say there exists an **incident** wave on **port 1** (i.e., $V_1^+(z_1) \neq 0$), while the incident waves on all other ports are known to be **zero** (i.e., $V_2^+(z_2) = V_3^+(z_3) = V_4^+(z_4) = 0$).

Say we measure/determine the voltage of the wave flowing **into port 1**, at the port 1 **plane** (i.e., determine $V_1^+(z_1 = z_{1P})$).
Say we then measure/determine the voltage of the wave flowing **out** of **port 2**, at the port 2 plane (i.e., determine $V_2^-(z_2 = z_{2P})$).

The complex ratio between $V_1^+(z_1 = z_{1P})$ and $V_2^-(z_2 = z_{2P})$ is know as the **scattering parameter** S_{21}:

$$S_{21} = \frac{V_2^-(z = z_2)}{V_1^+(z = z_1)} = \frac{V_{02}^- \, e^{+j\beta z_{2P}}}{V_{01}^+ \, e^{-j\beta z_{1P}}} = \frac{V_{02}^-}{V_{01}^+} e^{+j\beta(z_{2P}+z_{1P})}$$

Likewise, the scattering parameters S_{31} and S_{41} are:

$$S_{31} = \frac{V_3^-(z_3 = z_{3P})}{V_1^+(z_1 = z_{1P})} \qquad \text{and} \qquad S_{41} = \frac{V_4^-(z_4 = z_{4P})}{V_1^+(z_1 = z_{1P})}$$

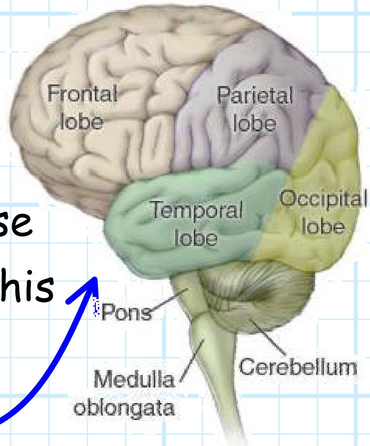
We of course could **also** define, say, scattering parameter S_{34} as the ratio between the complex values $V_4^+(z_4 = z_{4P})$ (the wave **into** port 4) and $V_3^-(z_3 = z_{3P})$ (the wave **out of** port 3), given that the input to all other ports (1,2, and 3) are zero.
Thus, more **generally**, the ratio of the wave incident on port *n* to the wave emerging from port *m* is:

$$S_{mn} = \frac{V_m^-(z_m = z_{mP})}{V_n^+(z_n = z_{nP})} \qquad \text{(given that} \quad V_k^+(z_k) = 0 \ \text{ for all } k \neq n\text{)}$$

Note that frequently the port positions are assigned a **zero** value (e.g., $z_{1P} = 0$, $z_{2P} = 0$). This of course **simplifies** the scattering parameter calculation:

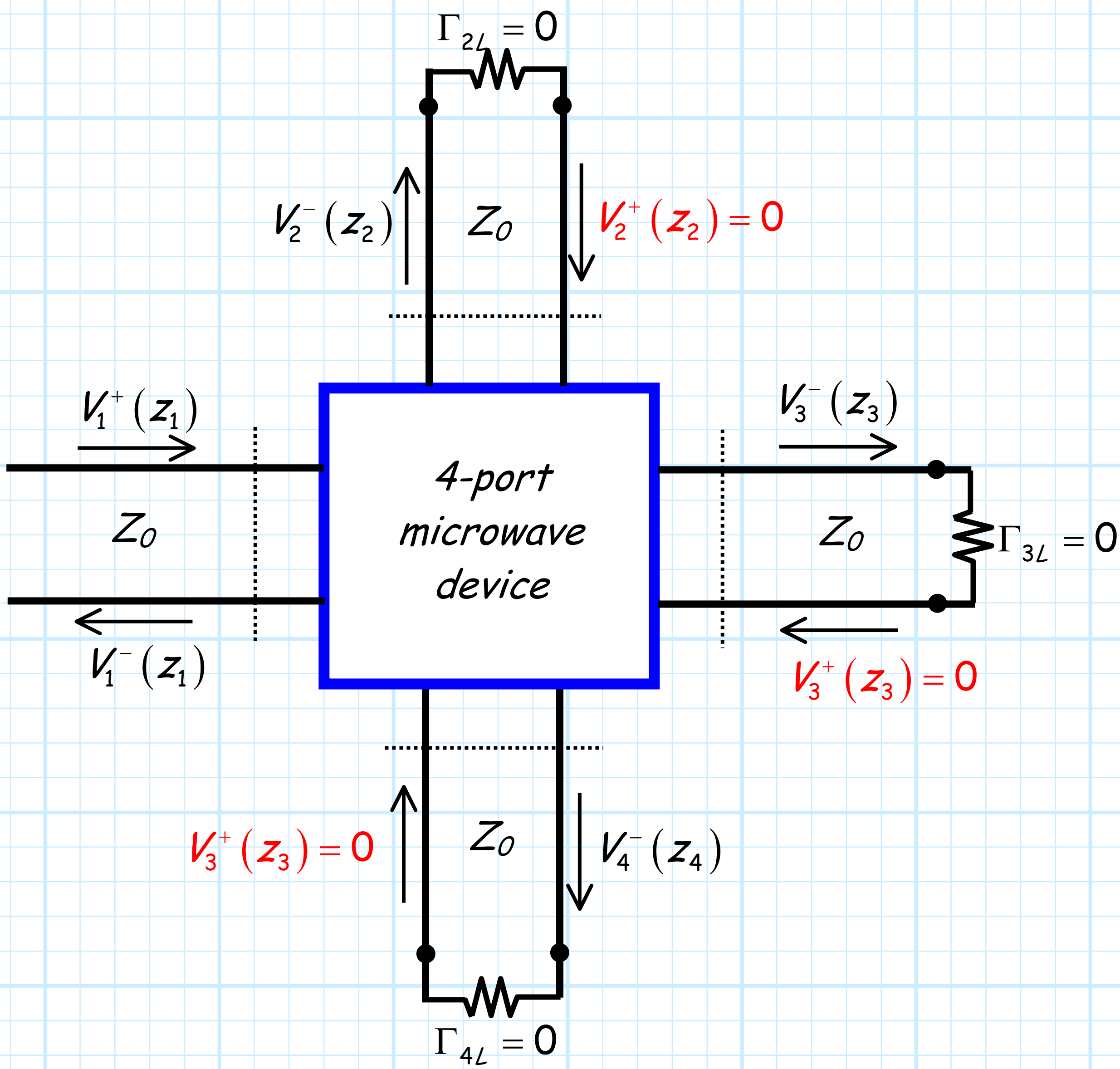
$$S_{mn} = \frac{V_m^-(z_m = 0)}{V_n^+(z_n = 0)} = \frac{V_{0m}^- \, e^{+j\beta 0}}{V_{0n}^+ \, e^{-j\beta 0}} = \frac{V_{0m}^-}{V_{0n}^+}$$

We will **generally assume** that the port locations are defined as $z_{nP} = 0$, and thus use the **above** notation. But **remember** where this expression came from!

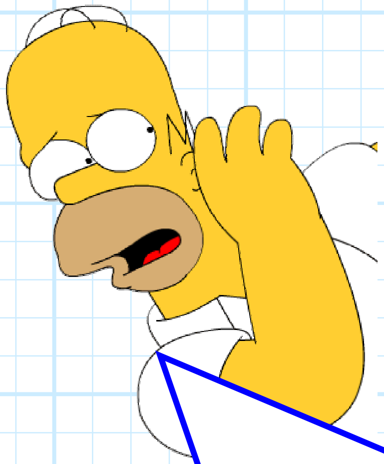


Q: *But how do we ensure that* ***only one*** *incident wave is non-zero ?*

A: Terminate all other ports with a **matched load**!



Note that **if** the ports are terminated in a **matched load** (i.e., $Z_L = Z_0$), then $\Gamma_{nL} = 0$ and therefore:



$$V_n^+(z_n) = 0$$

In other words, terminating a port ensures that there will be **no signal** incident on that port!

Q: *Just between you and me, I think you've messed this up!* ***In all*** *previous handouts you said that if* $\Gamma_L = 0$, the wave in the **minus** direction would be zero:

$$V^-(z) = 0 \quad \text{if} \quad \Gamma_L = 0$$

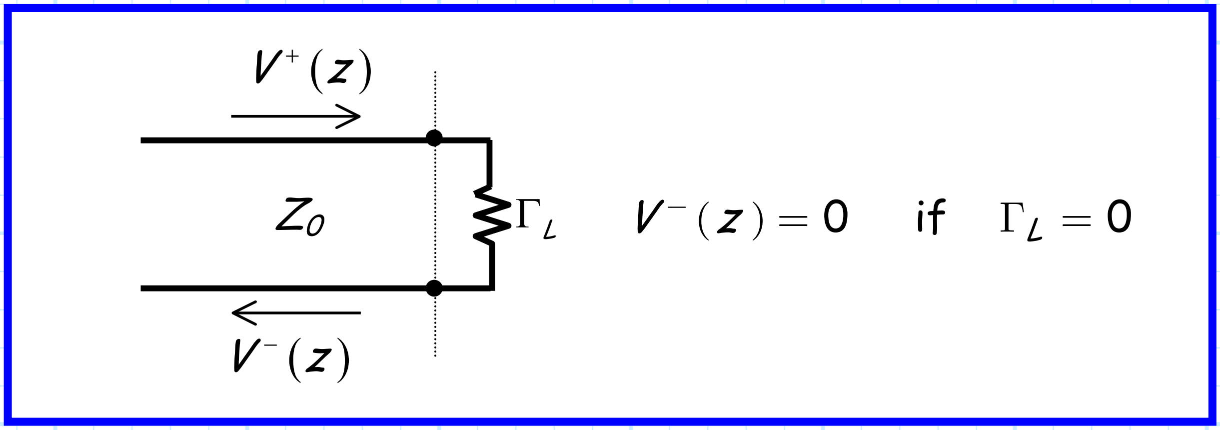
but just ***now*** *you said that the wave in the* ***positive*** *direction would be zero:*

$$V^+(z) = 0 \quad \text{if} \quad \Gamma_L = 0$$

Of course, there is ***no way*** *that* ***both*** *statements can be correct!*

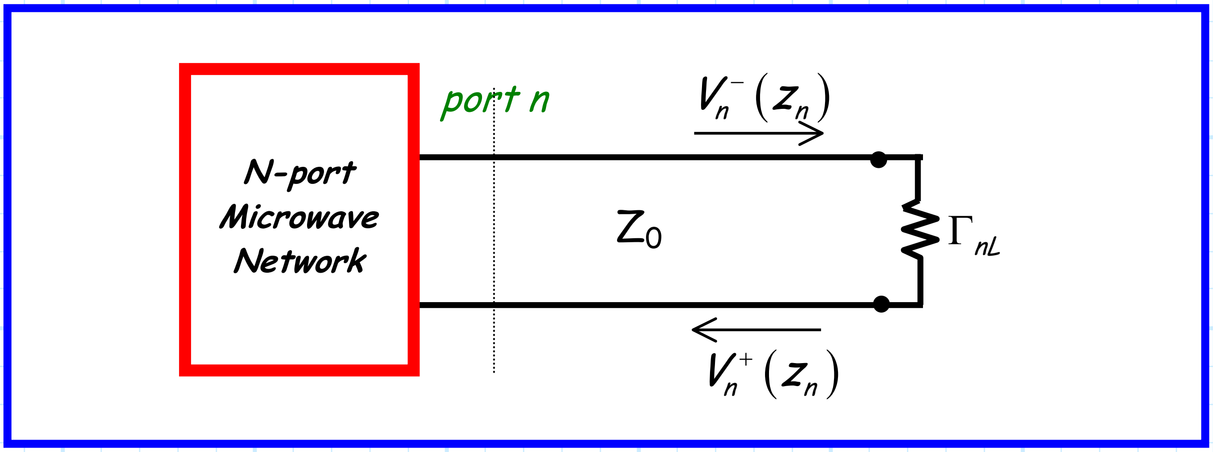
A: Actually, **both** statements are correct! You must be careful to understand the **physical definitions** of the plus and minus directions—in other words, the propagation directions of waves $V_n^+(z_n)$ and $V_n^-(z_n)$!

For example, we **originally** analyzed this case:



In this original case, the wave **incident** on the load is $V^+(z)$ (**plus** direction), while the **reflected** wave is $V^-(z)$ (**minus** direction).

Contrast this with the case we are **now** considering:

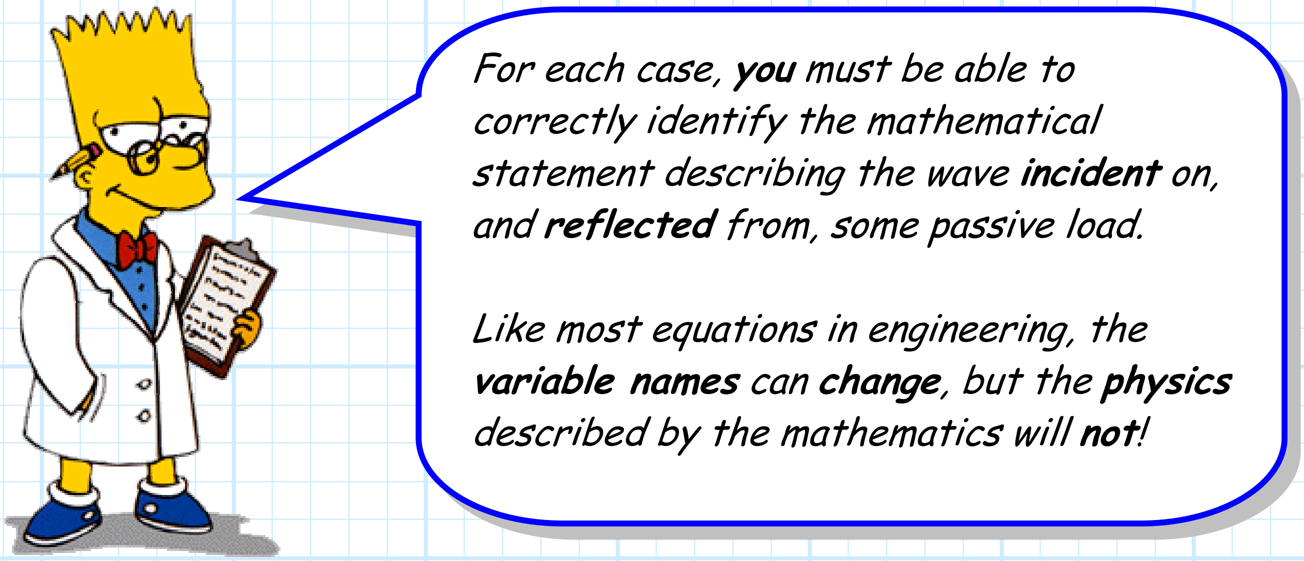


For this current case, the situation is **reversed**. The wave incident on the load is **now** denoted as $V_n^-(z_n)$ (coming **out** of port *n*), while the wave reflected off the load is **now** denoted as $V_n^+(z_n)$ (going **into** port *n*).

As a result, $V_n^+(z_n) = 0$ when $\Gamma_{nL} = 0$!

Perhaps we could more **generally** state that:

$$V^{reflected}(z = z_L) = \Gamma_L \, V^{incident}(z = z_L)$$



Now, **back** to our discussion of **S-parameters**. We found that **if** $z_{nP} = 0$ for all ports n, the scattering parameters could be directly written in terms of wave **amplitudes** V_{0n}^+ and V_{0m}^-.

$$S_{mn} = \frac{V_{0m}^-}{V_{0n}^+} \qquad \text{(given that} \quad V_k^+(z_k) = 0 \ \text{ for all } k \neq n)$$

Which we can now **equivalently** state as:

$$S_{mn} = \frac{V_{0m}^-}{V_{0n}^+} \qquad \text{(given that all ports, except port } n\text{, are \textbf{matched})}$$

One more **important** note—notice that for the **matched** ports (i.e., those ports with **no** incident wave), the voltage of the exiting **wave** is also the **total** voltage!

$$\begin{aligned} V_m(z_m) &= V_{0m}^{+}\, e^{-j\beta z_n} + V_{0m}^{-}\, e^{+j\beta z_n} \\ &= 0 + V_{0m}^{-}\, e^{+j\beta z_m} \\ &= V_{0m}^{-}\, e^{+j\beta z_m} \qquad \text{(for all terminated ports)} \end{aligned}$$

Thus, the value of the exiting wave **at** each terminated **port** is likewise the value of the total voltage **at** those ports:

$$V_m(0) = V_{0m}^{-} \qquad \text{(for all terminated ports)}$$

And so, we can express **some** of the scattering parameters equivalently as:

$$S_{mn} = \frac{V_m(0)}{V_{0n}^{+}} \qquad \text{(for \textbf{matched} port } m\text{, i.e., for } m \neq n\text{)}$$

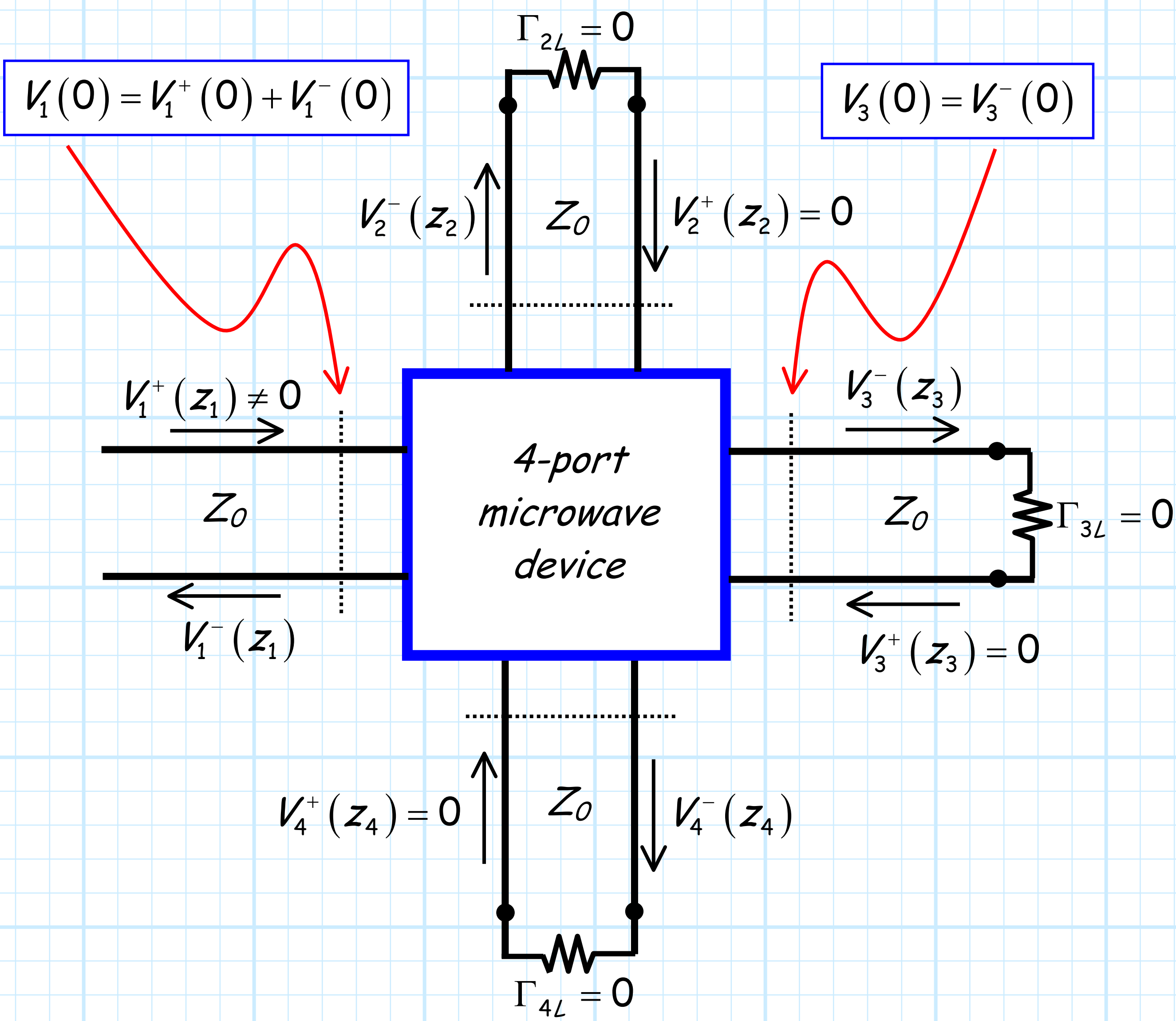
You might find this result **helpful** if attempting to determine scattering parameters where $m \neq n$ (e.g., S_{21}, S_{43}, S_{13}), as we can often use traditional **circuit theory** to easily determine the **total** port voltage $V_m(0)$.

However, we **cannot** use the expression above to determine the scattering parameters when $m = n$ (e.g., S_{11}, S_{22}, S_{33}).

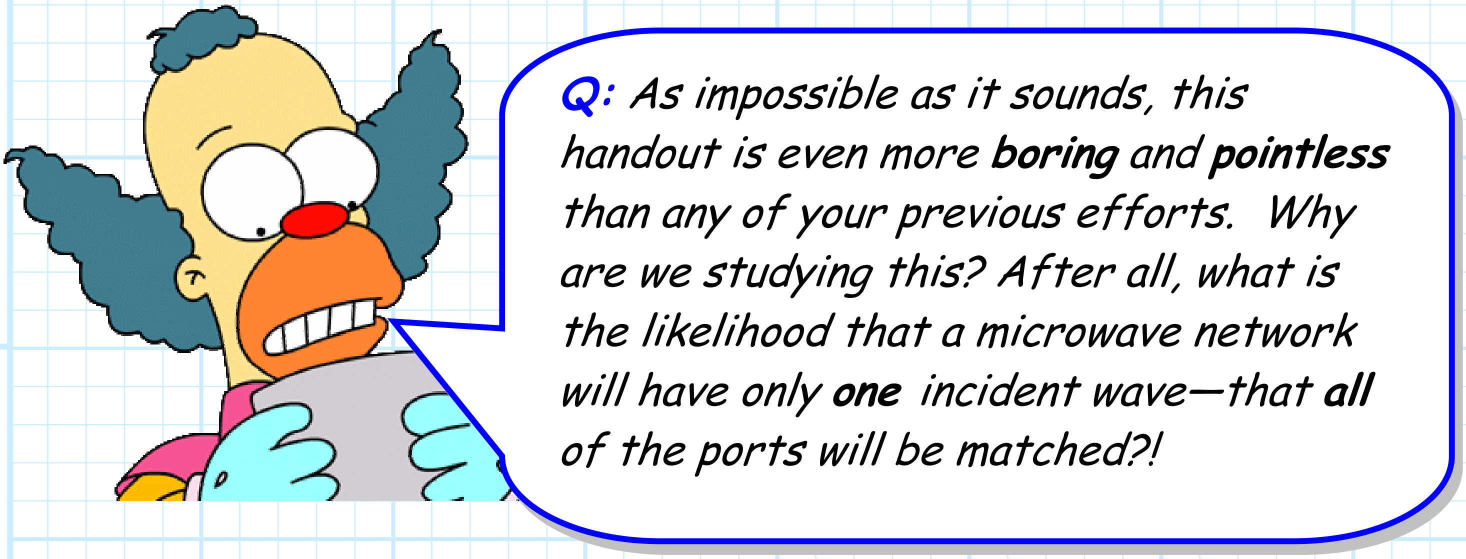
Think about this! The scattering parameters for these cases are:

$$S_{nn} = \frac{V_{0n}^-}{V_{0n}^+}$$

Therefore, port n is a port where there actually **is** some incident wave V_{0n}^+ (port n is **not** terminated in a matched load!). Thus, the total voltage is **not** simply the value of the exiting wave, as **both** an incident wave and exiting wave exists at port n.



Typically, it is **much** more difficult to determine/measure the scattering parameters of the form S_{nn}, as opposed to scattering parameters of the form S_{mn} (where $m \neq n$) where there is **only** an **exiting** wave from port m.



A: OK, say that our ports are **not** matched, such that we have waves **simultaneously** incident on **each** of the **four** ports of our device.

Since the device is **linear**, the output at any port due to **all** the incident waves is simply the coherent **sum** of the output at that port due to **each** wave!

For example, the output wave at port 3 can be determined by (assuming $z_{nP} = 0$):

$$V_{03}^- = S_{34} V_{04}^+ + S_{33} V_{03}^+ + S_{32} V_{02}^+ + S_{31} V_{01}^+$$

More generally, the output at port m of an N-port device is:

$$V_{0m}^- = \sum_{n=1}^{N} S_{mn} V_{0n}^+ \qquad (z_{nP} = 0)$$

This expression can be written in **matrix** form as:

$$\bar{\mathbf{V}}^{-} = \bar{\bar{\mathbf{S}}}\ \bar{\mathbf{V}}^{+}$$

Where $\bar{\mathbf{V}}^{-}$ is the **vector**:

$$\bar{\mathbf{V}}^{-} = \left[V_{01}^{-}, V_{02}^{-}, V_{03}^{-}, \ldots, V_{0N}^{-}\right]^{T}$$

and $\bar{\mathbf{V}}^{+}$ is the vector:

$$\bar{\mathbf{V}}^{+} = \left[V_{01}^{+}, V_{02}^{+}, V_{03}^{+}, \ldots, V_{0N}^{+}\right]^{T}$$

Therefore $\bar{\bar{\mathbf{S}}}$ is the **scattering matrix**:

$$\bar{\bar{\mathbf{S}}} = \begin{bmatrix} S_{11} & \cdots & S_{1n} \\ \vdots & \ddots & \vdots \\ S_{m1} & \cdots & S_{mn} \end{bmatrix}$$

The scattering matrix is a N by N matrix that **completely characterizes** a linear, N-port device. Effectively, the scattering matrix describes a multi-port device the way that Γ_L describes a single-port device (e.g., a load)!

But **beware**! The values of the scattering matrix for a particular device or network, just like Γ_L, are **frequency dependent**! Thus, it may be more instructive to **explicitly** write:

$$\bar{\bar{\mathbf{S}}}(\omega) = \begin{bmatrix} S_{11}(\omega) & \cdots & S_{1n}(\omega) \\ \vdots & \ddots & \vdots \\ S_{m1}(\omega) & \cdots & S_{mn}(\omega) \end{bmatrix}$$